Power (set) Description Logic

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Abstract. We explore the relationships between Description Logics and Set Theory. The study is carried on using, on the set-theoretic side, a very rudimentary axiomatic set theory Ω, consisting of only four axioms characterizing binary union, set difference, inclusion, and the power-set. The approach is then completed defining $\mathcal{ALC}^{\Omega}$, an extension of $\mathcal{ALC}$, in which concepts are naturally interpreted as sets living in Ω-models. In $\mathcal{ALC}^{\Omega}$ not only membership between concepts is allowed—even admitting circularity—but also the power-set construct is exploited to add metamodeling capabilities. We conclude providing a polynomial translation of $\mathcal{ALC}^{\Omega}$ in $\mathcal{ALCOI}$ and proving its basic traits, among which the validity of the finite model property.

1 Introduction

Concept and concept constructors in Description Logics (DLs) allow to manage information built-up and stored as collection of elements of a given domain. In this paper we would like to take the above statement seriously and put forward a DL doubly linked with a (very simple) set theory. Such a theory will be suitable to manipulate concepts as first-class citizens, in the sense that it will allow the possibility to have concepts as instances (a.k.a. elements) of other concepts. Actually, the idea of enhancing the language of description logics with statements of the form $C \in D$, with C and D concepts is not new, as assertions of the form $D(A)$, with A a concept name, are allowed in OWL-Full [20]. Here, while we do not consider roles as possible instances of concepts, we would like to push the approach a little forward, allowing not only the possibility of stating that an arbitrary concept C can be thought as an instance of another one ($C \in D$), but also opening up our view on two further directions:

1. the possibility of having a concept as an instance of itself: $C \in C$[1];
2. the possibility of talking about *all possible* sub-concepts of a given concept, adding a power-set construct $\texttt{Pow}(C)$.

In order to realize our plan we introduce a DL, to be dubbed $\mathcal{ALC}^{\Omega}$, whose two parents are $\mathcal{ALC}$ and a rudimentary (finitely axiomatized) set theory Ω.

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[1] Self membership is allowed for concept names in [16], by assertions of the form $a(a)$

Motik has shown in [16] that the semantics of metamodeling adopted in OWL-Full leads to undecidability already for $\mathcal{ALC}$-Full, due to the free mixing of logical and metalogical symbols. In [16], limiting this free mixing but allowing names to be interpreted as concepts and to occur as instances of other concepts, two alternative semantics (the Contextual π-semantics and the Hilog ν-semantics) are proposed for metamodeling and the decidability of $\mathcal{SHOIQ}$ extended with metamodeling is proved under either semantics.

Starting from [16], many other approaches to metamodeling have been proposed in the literature. Most of them [6, 11, 14, 9] are based on a Hilog semantics, while [19, 17] define extensions of OWL DL and of $\mathcal{SHIQ}$ (respectively), based on semantics interpreting concepts as well-founded sets. Here, we propose an extension of $\mathcal{ALC}$ with power-set and membership axioms among concepts, whose semantics is naturally defined using sets living in Ω-models (not necesarily well-founded). We show that the power-set construct in $\mathcal{ALC}^{\Omega}$ allows to capture in a natuaral way the interactions between concepts and metaconcepts.

We prove that $\mathcal{ALC}^{\Omega}$ is decidable by defining, for any $\mathcal{ALC}^{\Omega}$ knowledge base K, a polynomial translation K^T into $\mathcal{ALCOI}$, exploiting the correspondence studied in [5] between the membership relation in the set theory and a normal modality. We show that the translation K^T enjoys the finite model property and we exploit it in the proof of completeness of the translation. From the translation in $\mathcal{ALCOI}$ we also get an Ex-PTIME upper bound on the complexity of satisfiability in $\mathcal{ALC}^{\Omega}$. Interestingly enough, our translation has strong relations with the first-order reduction used in [8, 11, 14].

2   Preliminaries

2.1   The description logics $\mathcal{ALC}$ and $\mathcal{ALCOI}$

Let N_C be a set of concept names, N_R a set of role names and N_I a set of individual names. The set $\mathcal{C}$ of $\mathcal{ALC}$ *concepts* can be defined inductively as follows:

- $A \in N_C$, $\top$ and $\bot$ are *concepts* in $\mathcal{C}$;
- if $C, D \in \mathcal{C}$ and $R \in N_R$, then $C \sqcap D, C \sqcup D, \neg C, \forall R.C, \exists R.C$ are *concepts* in $\mathcal{C}$

A knowledge base (KB) K is a pair $(\mathcal{T}, \mathcal{A})$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox. The TBox $\mathcal{T}$ is a set of concept inclusions (or subsumptions) $C \sqsubseteq D$, where C, D are concepts in $\mathcal{C}$. The ABox $\mathcal{A}$ is a set of assertions of the form $C(a)$ and $R(a, b)$ where C is a concept, $R \in N_R$, and $a, b \in N_I$.

An interpretation for $\mathcal{ALC}$ [2] is a pair $I = \langle \Delta, \cdot^I \rangle$ where:

- Δ is a domain, whose elements are denoted by $x, y, z, \ldots$;
- $\cdot^I$ is an extension function that maps each concept name $C \in N_C$ to a set $C^I \subseteq \Delta$, each role name $R \in N_R$ to a binary relation $R^I \subseteq \Delta \times \Delta$, and each individual name $a \in N_I$ to an element $a^I \in \Delta$; the function $\cdot^I$ is extended to complex concepts as follows:
$$\top^I = \Delta$$
$$\bot^I = \emptyset$$

$$(\neg C)^I = \Delta \backslash C^I$$
$$(C \sqcap D)^I = C^I \cap D^I$$
$$(C \sqcup D)^I = C^I \cup D^I$$
$$(\forall R.C)^I = \{x \in \Delta \mid \forall y.(x,y) \in R^I \rightarrow y \in C^I\}$$
$$(\exists R.C)^I = \{x \in \Delta \mid \exists y.(x,y) \in R^I \text{ and } y \in C^I\}$$

The notion of satisfiability of a KB in an interpretation is defined as follows:

Definition 1 (Satisfiability and entailment). *Given an $\mathcal{ALC}$ interpretation $I = \langle \Delta, \cdot^I \rangle$:*
- I satisfies an inclusion $C \sqsubseteq D$ if $C^I \subseteq D^I$;
- I satisfies an assertion $C(a)$ if $a^I \in C^I$;
- I satisfies an assertion $R(a,b)$ if $(a^I, b^I) \in R^I$.
Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$, an interpretation I satisfies $\mathcal{T}$ (resp. $\mathcal{A}$) if I satisfies all inclusions in $\mathcal{T}$ (resp. all assertions in $\mathcal{A}$); I is a model of K if I satisfies $\mathcal{T}$ and $\mathcal{A}$.
Let a query *F be either an inclusion $C \sqsubseteq D$ (where C and D are concepts) or an assertion $C(a)$. F is entailed by K, written $K \models F$, if for all models $I = \langle \Delta, \cdot^I \rangle$ of K, I satisfies F.*

Given a knowledge base K, the *subsumption* problem is the problem of deciding whether an inclusion $C \sqsubseteq D$ is entailed by K. The *instance checking* problem is the problem of deciding whether an assertion $C(a)$ is entailed by K. The *concept satisfiability* problem is the problem of deciding, for a concept C, whether C is consistent with K (i.e., there is a model I of K, such that $C^I \neq \emptyset$).

In the following we will also consider the description logic $\mathcal{ALCOI}$ allowing inverse roles and nominals. For a role $R \in N_R$, its *inverse* is a role, denoted by R^-, which can be used in existential and universal restrictions with the following semantics: $(x,y) \in (R^-)^I$ if and only if $(y,x) \in R^I$. For a named individual $a \in N_I$, the *nominal* $\{a\}$ is a concept with the following semantics: $(\{a\})^I = \{a^I\}$.

2.2 The theory Ω

The first-order theory Ω consists of the following four axioms in the language with relational symbols $\in$ and $\subseteq$, and functional symbols $\cup, \backslash, Pow$:

$$x \in y \cup z \leftrightarrow x \in y \vee x \in z;$$
$$x \in y \backslash z \leftrightarrow x \in y \wedge x \notin z;$$
$$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y);$$
$$x \in Pow(y) \leftrightarrow x \subseteq y.$$

In an Ω-model *everything* is supposed to be a set. Hence, a set will have (only) sets as its elements and circular definition of sets are allowed (such as a set admitting itself as one of its elements) are *not* forbidden. Moreover, not postulating in Ω any *link* between membership $\in$ and equality—in axiomatic terms, having no *extensionality* (axiom)—Ω-models in which there are different sets with equal collection of elements, are admissible.

The most natural Ω-model—in which extensionally different sets are, in fact, always different— is the collection of well-founded sets $\mathsf{HF} = \mathsf{HF}^0 = \bigcup_{n \in \mathbb{N}} \mathsf{HF}_n$, where:

$$\mathsf{HF}_0 = \emptyset;$$
$$\mathsf{HF}_{n+1} = Pow(\mathsf{HF}_n).$$

In HF^0 every system of set-theoretic equations of the form:

$$\begin{cases} x_1 = \{x_{1,1}, \ldots, x_{1,m_1}\}; \\ x_2 = \{x_{2,1}, \ldots, x_{2,m_2}\}; \\ \vdots \qquad\qquad \vdots \\ x_n = \{x_{n,1}, \ldots, x_{n,m_n}\}, \end{cases}$$

where $x_{i,j}$ for $j = 1, \ldots, m_i$ is one among $x_1, \ldots x_{i-1}$, finds a unique solution.

If we drop the index-ordering restriction on variables appearing in the right-hand-side of set-theoretic equations (e.g allowing equations such as $x = \{x\}$), in order to guarantee the existence of solutions in the model we need to work with larger universes. The most natural (and minimal) among them is a close relative of HF^0 and goes under the name of $\mathsf{HF}^{1/2}$ (see [1, 18]).

Finally, a further enrichment of both HF^0 and $\mathsf{HF}^{1/2}$ is obtained by adding *atoms*, that is copies of the empty-set, to be denoted by $\mathbf{a}_1, \mathbf{a}_2, \ldots$ and collectively represented by $\mathbb{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$. The resulting universes will be denoted by $\mathsf{HF}^0(\mathbb{A})$ and $\mathsf{HF}^{1/2}(\mathbb{A})$.

A complete discussion relative to universes of sets to be used as models of Ω goes beyond the scope of this paper. However, it is convenient to point out that, in all cases of interest for us here, an especially simple view of Ω-models can be given using *finite graphs*. Actually, HF^0 or $\mathsf{HF}^{1/2}$ can be seen as the collection of finite (either acyclic or cyclic) graphs where sets are represented by nodes and arcs depict the membership relation among sets (see [18]). Given one such *membership* graph G it is useful to single out a special node (the *point* of the graph), to isolate the specific set for which the description is introduced.

In the next section, we will regard the domain Δ of a DL interpretation as a fragment of the universe of an Ω-model, i.e. Δ will be regarded as a set of sets of the theory Ω rather than as a set of individuals, as usual in description logics.

3   The description logic $\mathcal{ALC}^\Omega$

We start from the observation that in $\mathcal{ALC}$ concepts are interpreted as sets (namely, sets of domain elements) and we generalize $\mathcal{ALC}$ by allowing concepts to be interpreted as sets of the set theory Ω. In addition, we extend the language of $\mathcal{ALC}$ by introducing the power-set constructor as a new concept constructor, and by allowing membership relations among concepts to occur in the knowledge base. We call $\mathcal{ALC}^\Omega$ the resulting extension of $\mathcal{ALC}$.

As before, let N_I , N_C, and N_R be the set of individual names, concept names, and role names in the language, respectively. In building complex concepts, in addition

to the constructs of $\mathcal{ALC}$, we also consider the difference $\backslash$ and the power-set $\mathtt{Pow}$ constructs.

The set of $\mathcal{ALC}^{\Omega}$ *concepts* are defined inductively as follows:

- $A \in N_C$, $\top$ and $\bot$ are $\mathcal{ALC}^{\Omega}$ *concepts*;
- if C, D are $\mathcal{ALC}^{\Omega}$ concepts and $R \in R_C$, then

$$C \sqcap D, C \sqcup D, \neg C, C \backslash D, \mathtt{Pow}(C), \forall R.C, \exists R.C$$

are $\mathcal{ALC}^{\Omega}$ *concepts*.

While the concept $C \backslash D$ can be easily defined as $C \sqcup \neg D$ in $\mathcal{ALC}$, this is not the case for the concept $\mathtt{Pow}(C)$. Informally, the instances of concept $\mathtt{Pow}(C)$ are all the subsets of the instances of concept C.

Besides ABox assertions of the form $C(a)$ with $a \in N_I$, we allow in the ABox *concept membership axioms* and *role membership axioms*, respectively, of the form:

$$C \in D \ \text{ and } \ (C, D) \in R,$$

where C and D are $\mathcal{ALC}^{\Omega}$ concepts and R is a role name.

Considering an example taken from [22, 16], in the language of $\mathcal{ALC}^{\Omega}$ we can represent the fact that eagles are in the red list of endangered species, by the axiom $Eagle \in RedListSpecies$, and that Harry is an eagle, by the assertion $Eagle(harry)$. Observe that nothing prevents us from having a concept which includes itself as an instance, as specified by the axiom $C \in C$. Using self-membership we sgnificantly gain in expressivity: consider the concept $notModifiableList$, consisting of those list that cannot be modified (if not by, say, a specifically enforced law); certainly, for example, it would be reasonable to ask $RedListSpecies \in notModifiableList$ but, more interestingly, we would also clearly want $notModifiableList \in notModifiableList$. Observe also that the additional expressivity of the language, in which general concepts (and not only concept names) can be instances of other concepts, allows for instance to represent the fact that polar bears are in the red list of endangered species, by the axiom $Polar \sqcap Bear \in RedListSpecies$. We can further represent the fact the polar bears are more endangered than eagles by adding a role $moreEndangered$ and the role membership axiom $(Polar \sqcap Bear, Eagle) \in moreEndangered$.

The presence of the power-set constructor is useful to represent statements such as "all the instances of species in the Red List are not allowed to be hunted". By the concept inclusion axiom:

$$RedListSpecies \sqsubseteq \mathtt{Pow}(CannotHunt),$$

all the instances of classes in the $RedListSpecies$ (such as $Eagle$) cannot be hunted. As shown in [16], the meaning of the sentence above could be captured by combining the ν-semantics with SWRL [12], but not by the ν-semantics alone.

We define a semantics for $\mathcal{ALC}^{\Omega}$ by extending the $\mathcal{ALC}$ semantics in Section 2.1 to capture the meaning of concepts (including concept $\mathtt{Pow}(C)$) as elements (sets) of the domain Δ, chosen as a *transitive* set (i.e. a set x satisfying $(\forall y \in x)(\forall z \in y)(z \in x)$) in a model of Ω. Roles are interpreted as binary relations over the domain Δ. Individual names are interpreted as elements of a set of atoms $\mathbb{A}$ from which the sets in Δ are built.

Definition 2. *An interpretation for $\mathcal{ALC}^{\Omega}$ is a pair $I = \langle \Delta, \cdot^I \rangle$ over a set of atoms $\mathbb{A}$ where:*

- *the non-empty domain Δ is a transitive set chosen in a model $\mathcal{M}$ of Ω over the atoms in $\mathbb{A}$;[2]*
- *the extension function $\cdot^I$ maps each concept name $A \in N_C$ to an element $A^I \in \Delta$; each role name $R \in N_R$ to a binary relation $R^I \subseteq \Delta \times \Delta$; and each individual name $a \in N_I$ to an element $a^I \in \mathbb{A} \subseteq \Delta$.*

The function $\cdot^I$ is extended to complex concepts of $\mathcal{ALC}^{\Omega}$ as follows:

$$\top^I = \Delta$$
$$\bot^I = \emptyset$$
$$(\neg C)^I = \Delta \backslash C^I$$
$$(C \sqcap D)^I = (C^I \cap D^I) \cap \Delta$$
$$(C \sqcup D)^I = (C^I \cup D^I) \cap \Delta$$
$$(C \backslash D)^I = (C^I \backslash D^I) \cap \Delta$$
$$(Pow(C))^I = Pow(C^I) \cap \Delta$$
$$(\forall R.C)^I = \{x \in \Delta \mid \forall y((x,y) \in R^I \rightarrow y \in C^I)\}$$
$$(\exists R.C)^I = \{x \in \Delta \mid \exists y((x,y) \in R^I \land y \in C^I)\}$$

The semantics of usual DL concept constructs is defined as the standard one, save for the fact that Δ (which is a fragment of the domain of a model of Ω) is not guaranteed to be closed under union, intersection, etc. This is the reason why the intersection with Δ is to be added (on the right hand side of each item) when defining the interpretation of complex concepts.

Given an interpretation I, the satisfiability of inclusions and assertions is defined as in an $\mathcal{ALC}$ interpretation (Definition 1). Satisfiability of (concept and role) membership axioms in an interpretation I is defined as follows: I *satisfies* $C \in D$ if $C^I \in D^I$; I *satisfies* $(C, D) \in R$ if $(C^I, D^I) \in R^I$. With this addition, the notions of satisfiability of a KB and of entailment in $\mathcal{ALC}^{\Omega}$ (denoted by $\models_{\mathcal{ALC}^{\Omega}}$) can be defined as in Section 2.1.

The problem of instance checking in $\mathcal{ALC}^{\Omega}$ includes both the problem of verifying whether an assertion $C(a)$ is a logical consequence of the KB and the problem of verifying whether a membership $C \in D$ is a logical consequence of the KB (i.e., whether C is an instance of D).

In the next section, we define a polynomial encoding of the language $\mathcal{ALC}^{\Omega}$ into the description logic $\mathcal{ALCOI}$.

4   Translation of $\mathcal{ALC}^{\Omega}$ into $\mathcal{ALCOI}$

We define a translation of the logic $\mathcal{ALC}^{\Omega}$ into the description logic $\mathcal{ALCOI}$, including inverse roles and nominals. In [5] the membership relation $\in$ is used to represent a

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[2] In the following, for readability, we will denote by $\in$, Pow, $\cup$, $\backslash$ (rather than $Pow^{\mathcal{M}}$, $\cup^{\mathcal{M}}$, $\backslash^{\mathcal{M}}$) the interpretation in a model $\mathcal{M}$ of the predicate and function symbols $\in$, Pow, $\cup$, $\backslash$, respectively.

normal modality R. In this section, vice-versa, we exploit the correspondence between $\in$ and the accessibility relation of a modality by introducing a new (reserved) role e in N_R to represent the inverse of the membership relation: in any interpretation I, $(x, y) \in e^I$ will stand for $y \in x$. The idea underlying the translation is that each element u of the domain Δ in an $\mathcal{ALCOI}$ interpretation $I = \langle \Delta, \cdot^I \rangle$ can be regarded as the set of all the elements v such that $(u, v) \in e^I$.

The translation of a knowledge base $K = (\mathcal{T}, \mathcal{A})$ of $\mathcal{ALC}^\Omega$ into $\mathcal{ALCOI}$ can be defined as follows. First, we associate each concept C of $\mathcal{ALC}^\Omega$ to a concept C^T of $\mathcal{ALCOI}$ by replacing all occurrences of the power-set constructor Pow with a concept involving the universal restriction $\forall e$ (see below). More formally, we (inductively) define the translated C^T of C by simply recursively replacing every subconcept $\mathrm{Pow}(D)$ appearing in C by $\forall e.D^T$, while the translation T commutes with concept constructors in all other cases, as follows:

$\top^T = \top, \perp^T = \perp$
$B^T = B$, for $B \in N_C$
$(\neg C)^T = \neg C^T$
$(C_1 \sqcap C_2)^T = C_1^T \sqcap C_2^T$
$(C_1 \sqcup C_2)^T = C_1^T \sqcup C_2^T$
$(C \backslash D)^T = C^T \sqcap \neg D^T$
$\mathrm{Pow}(D) = \forall e.D^T$
$(\exists R.D)^T = \exists R.D^T$
$(\forall R.D)^T = \forall R.D^T$

Semantically this will result in interpreting any (sub)concept $(\mathrm{Pow}(D))^I$ by

$$(\forall e.D)^I = \{x \mid \forall y((x, y) \in e^I \rightarrow y \in D^I)\},$$

which, recalling that $(x, y) \in e^I$ stands for $y \in x$, will characterize the collection of (admissible) subsets of D^I:

$$(\forall e.D)^I = \{x \in \Delta \mid \forall y(y \in x \rightarrow y \in D^I)\},$$

that is,

$$(\forall e.D)^I = \{x \in \Delta \mid x \subseteq D^I\} = \mathrm{Pow}(D^I) \cap \Delta = (\mathrm{Pow}(D))^I,$$

as expected.

4.1 Translating TBox, ABox, and queries

We define a new TBox, $\mathcal{T}^T$, by introducing, for each inclusion $C \sqsubseteq D$ in $\mathcal{T}$, the inclusion $C^T \sqsubseteq D^T$ in $\mathcal{T}^T$. Additionally, for each (complex) concept C occurring in the knowledge base K (or in the query) on the l.h.s. of a membership axiom $C \in D$ or $(C, D) \in R$, we extend N_I with *a new individual name*[3] e_C and we add the concept equivalence:

$$C \equiv \exists e^-.\{e_C\} \tag{1}$$

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[3] The symbol e_C should remind the e-xtension of C.

in $\mathcal{T}^T$. From now on, new individual names such as e_C will be called *concept individual names*. This equivalence is intended to capture the property that, in all the models $I = \langle \Delta, \cdot^I \rangle$ of K^T, e_C^I is in relation e^I with all and only the instances of concept C, i.e., for all $y \in \Delta$, $(e_C^I, y) \in e^I$ if and only if $y \in C^I$.

As in the case of the power-set constructor, this fact can be verified by analyzing the semantics of $\exists e^-.\{e_C\}$:

$$(\exists e^-.\{e_C\})^I = \{x \in \Delta \mid \exists y((x,y) \in (e^-)^I \wedge y \in (\{e_C\})^I\},$$

which, recalling that e stands for $\ni$ and interpreting the nominal, will stand for

$$(\exists e^-.\{e_C\})^I = \{x \in \Delta \mid \exists y(x \in y \wedge y \in \{e_C^I\}\} = \{x \in \Delta \mid x \in e_C^I\},$$

which, by the concept equivalence $C \equiv \exists e^-.\{e_C\}$, is as to say that e_C^I and C^I have the same extension.

Remark 1. It is important to notice that every concept individual name of the sort e_C introduced above—that is, every individual name whose purpose is that of providing a name to the extension of C^I—, in general turns out to be in relation e with other elements of the domain Δ of I (unless C is an inconsistent concept and its extension is empty). This is in contrast with the assumption relative to other "standard" individual names $a \in N_I$, for which we will require $(\neg \exists e.\top)(a)$ (see below).

We define $\mathcal{A}^T$ as the set of assertions containing:

- for each concept membership axiom $C \in D$ in $\mathcal{A}$, the assertion $D^T(e_C)$,
- for each role membership axiom $(C, D) \in R$ in $\mathcal{A}$, the assertion $R(e_C, e_D)$,
- for each assertion $D(a)$ in $\mathcal{A}$, the assertion $D^T(a)$,
- for each assertion $R(a, b)$ in $\mathcal{A}$, the assertion $R(a, b)$ and, finally,
- for each (standard) individual name $a \in N_I$, the assertion $(\neg \exists e.\top)(a)$.

As noticed above, the last requirement forces all named individuals (in the language of the initial knowledge base K) to be interpreted as domain elements which are not in relation e with any other element.

Let $K^T = (\mathcal{T}^T, \mathcal{A}^T)$ be the knowledge base obtained by translating K into $\mathcal{ALCOI}$.

Example 1. Let $K = (\mathcal{T}, \mathcal{A})$ be the knowledge base considered above:
$\quad \mathcal{T} = \{RedListSpecies \sqsubseteq \mathbb{P}\mathrm{ow}(CannotHunt)\}$ and
$\quad \mathcal{A} = \{Eagle(harry), Eagle \in RedListSpecies, Polar \sqcap Bear \in RedListSpecies\}$.
By the translation above, we obtain:
$\quad \mathcal{T}^T = \{RedListSpecies \sqsubseteq \forall e.CannotHunt,,$
$\quad\quad\quad Eagle \equiv \exists e^-.\{e_{Eagle}\}, Polar \sqcap Bear \equiv \exists e^-.\{e_{Polar \sqcap Bear}\}\}$
$\quad \mathcal{A}^T = \{Eagle(harry), RedListSpecies(e_{Eagle}), RedListSpecies(e_{Polar \sqcap Bear})\},$

K^T entails $CannotHunt(harry)$ in $\mathcal{ALCOI}$. In fact, from $RedListSpecies(e_{Eagle})$ and $RedListSpecies \sqsubseteq \forall e.CannotHunt$, it follows that, in all models of K^T, $e_{Eagle}^I \in (\forall e.CannotHunt)^I$. Furthermore, from $Eagle \equiv \exists e^-.\{e_{Eagle}\}$ and the assertion $Eagle(harry)$, it follows that $(e_{Eagle}^I, harry^I) \in e^I$ holds. Hence, $harry^I \in CannotHunt^I$. As this holds in all models of K^T, $CannotHunt(harry)$ is a logical consequence of K^T. It is easy to see that $Eagle \sqsubseteq CannotHunt$ follows from K^T as well.

Let F be a *query* of the form $C \sqsubseteq D$, $C(a)$ or $C \in D$. We assume that all the individual names, concept names and role names occurring in F also occur in K and we define a translation F^T of the query F as follows:

- if F is a subsumption $C \sqsubseteq D$, then F^T is the subsumption $C^T \sqsubseteq D^T$;
- if F is an assertion $C(a)$, then F^T is the assertion $C^T(a)$;
- if F is a membership axiom $C \in D$, then F^T is the assertion $D^T(e_C)$.

In the following we prove the soundness and completeness of the translation of an $\mathcal{ALC}^{\Omega}$ knowledge base into $\mathcal{ALCOI}$.

Proposition 1 (Soundness of the translation). *The translation of an $\mathcal{ALC}^{\Omega}$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ into $\mathcal{ALCOI}$ is sound, that is, for any query F:*

$$K^T \models_{\mathcal{ALCOI}} F^T \Rightarrow K \models_{\mathcal{ALC}^{\Omega}} F.$$

Proof. By contraposition, assume $K \not\models_{\mathcal{ALC}^{\Omega}} F$ and let $I = \langle \Delta, \cdot^I \rangle$ be a model of K in $\mathcal{ALC}^{\Omega}$ that falsifies F[4]. We build an $\mathcal{ALCOI}$ interpretation $I' = \langle \Delta', \cdot^{I'} \rangle$, which is going to be a model of K^T falsifying F in $\mathcal{ALCOI}$, by letting:

- $\Delta' = \Delta$;
- for all $B \in N_C$, $B^{I'} = B^I$;
- for all roles $R_j \in N_R$, $R_j^{I'} = R_j^I$;
- for all $x, y \in \Delta'$, $(x, y) \in e^{I'}$ if and only if $y \in x$;
- for all (standard) individual name $a \in N_I$, $a^{I'} = a^I \in \mathbb{A}$;
- for all $e_C \in N_I$, $e_C^{I'} = C^I \in \Delta'$;

Observe that the set $C^{I'}$ is a subset of Δ', as $\Delta' = \Delta$ is a transitive set and therefore it includes as a subset each element in Δ. Also, each element $a^I \in \mathbb{A}$ belongs to Δ and hence to Δ'.

We can prove by induction on the structure of concepts that, for all $x \in \Delta'$,

$$x \in (C^T)^{I'} \text{ if and only if } x \in C^I \tag{2}$$

For the base case, the property above holds by definition for all concepts names in N_C, as well as for $C = \top$ and $C = \bot$ (given that $\top^T = \top$ and $\bot^T = \bot$).

For the inductive step, let $C = C_1 \sqcap C_2$ and let $x \in ((C_1 \sqcap C_2)^T)^{I'} = (C_1^T \sqcap C_2^T)^{I'}$, for some $x \in \Delta'$. As I' is an $\mathcal{ALCOI}$ interpretation, $x \in (C_1^T)^{I'}$ and $x \in (C_2^T)^{I'}$ and since by induction (2) holds for concepts C_1 and C_2, we have $x \in C_1^I$ and $x \in C_2^I$. Therefore, $x \in (C_1^I \cap C_2^I) \cap \Delta$—since $x \in \Delta' = \Delta$—from which, by definition $\mathcal{ALC}^{\Omega}$ interpretation, $x \in (C_1 \sqcap C_2)^I$. It is easy to see that the vice-versa also holds, i.e., if $x \in (C_1 \sqcap C_2)^I$ then $x \in ((C_1 \sqcap C_2)^T)^{I'}$.

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[4] Observe that, on the grounds of Proposition 2 below, we could safely assume the domain Δ to be *essentially* a transitive set in the standard model $\mathsf{HF}^{1/2}(\mathbb{A})$ of Ω, (i.e. the collection of hereditarily finite rational hypersets over atoms in $\mathbb{A}$). As a matter of fact, the domain Δ is (a graph) obtained duplicating sets (nodes) representing extensionally equal but pairwise distinct sets/elements.

For the case $C = \text{Pow}(D)$, let $x \in ((\text{Pow}(D))^T)^{I'} = (\forall e.D^T)^{I'}$, for some $x \in \Delta'$. As I' is an $\mathcal{ALCOI}$ interpretation, for all $y \in \Delta'$, if $(x,y) \in e^{I'}$ then $y \in (D^T)^{I'}$. By construction of I', $(x,y) \in e^{I'}$ if and only if $y \in x$ and, by inductive hypothesis, $y \in (D^T)^{I'}$ if and only if $y \in D^I$. Hence, in I, for all $y \in \Delta$ such that $y \in x$, $y \in D^I$. Therefore, $x \subseteq D^I$, that is $x \in Pow(D^I)$. Moreover, as $x \in \Delta$, $x \in Pow(D^I) \cap \Delta$, and therefore, by definition of $\mathcal{ALC}^{\Omega}$ interpretation, $x \in (\text{Pow}(D))^I$. The vice-versa can be proved similarly.

Let us consider the case $C = \exists R.D$. Let $x \in ((\exists R.D)^T)^{I'}$ for some $x \in \Delta'$. As $x \in (\exists R.D^T)^{I'}$ and I' is an $\mathcal{ALCOI}$ interpretation, there is a $y \in \Delta'$ such that $(x,y) \in R^{I'}$ and $y \in D^{I'}$. By inductive hypothesis, $y \in D^I$. Furthermore, by construction of I', it must be that $(x,y) \in R^I$ and $x \in \Delta$. Hence, $x \in (\exists R.D)^I$. The vice-versa can be proved similarly as well as all the other cases for the concept C.

All the other cases are proved in a similar way.

Using (2) we can now check that all the axioms and assertions in K^T hold in I'.

For an inclusion axiom $C^T \sqsubseteq D^T \in \mathcal{T}^T$, the corresponding inclusion axiom $C \sqsubseteq D$ is in $\mathcal{T}$. If $x \in (C^T)^{I'}$ for some $x \in \Delta'$, by (2) $x \in C^I$ and, by the inclusion $C \sqsubseteq D \in \mathcal{T}$, $x \in D^I$. Hence, again by (2), $x \in (D^T)^{I'}$.

For each assertion $D^T(a) \in \mathcal{A}^T$, obtained from the translation of the assertion $D(a) \in \mathcal{A}$. From the fact that $D(a)$ is satisfied by I, i.e. $a^I \in D^I$, given property (2), it follows that $a^{I'} = a^I \in (D^T)^{I'}$.

For each assertion $D^T(e_C) \in \mathcal{A}^T$ obtained from the translation of a membership axiom $C \in D$, from the fact that I is a model of K, we know that $C^I \in D^I$ holds. We want to show that $e_C^{I'} \in (D^T)^{I'}$. As, by construction, $e_C^{I'} = C^I \in \Delta = \Delta'$, from $C^I \in D^I$, it follows that $e_C^{I'} \in D^I$. By property (2), $e_C^{I'} \in (D^T)^{I'}$.

For each assertion $R(e_C, e_D) \in \mathcal{A}^T$ obtained from the translation of a role membership axiom $(C, D) \in R$, from the fact that I is a model of K, we know that $(C^I, D^I) \in R^I$ holds. We want to show that $(e_C^{I'}, e_D^{I'}) \in R^{I'}$. As, by construction, $e_C^{I'} = C^I \in \Delta = \Delta'$ and $e_D^{I'} = D^I \in \Delta = \Delta'$ from $(C^I, D^I) \in R^I$, it follows that $(e_C^{I'}, e_D^{I'}) \in R^I$. By the definition of role interpretation in I', $(e_C^{I'}, e_D^{I'}) \in R^{I'}$.

For each assertion $(\neg \exists e.\top)(a)$, for $a \in N_I$, it is easy to see that $a^{I'} \notin (\exists e.\top)^{I'}$, as $a^{I'} = a^I \in \mathbb{A}$ and an element of $\mathbb{A}$ in Δ can be regarded as an empty set. As there is no $y \in a^I$, by construction of the model I', there is no y such that $(a^{I'}, y) \in e^{I'}$.

In a similar way it is easy to show that, for any query F, F^T is satisfied in I' if and only if F is satisfied in I.

To conclude the proof, we still need to show that the axiom $C^T \equiv \exists e^-.\{e_C\}$ is satisfied in I' for all the concepts C occurring in K on the l.h.s. of membership axioms. Let $x \in (C^T)^{I'}$. By property (2), $x \in C^I$ and, by construction, $e_C^{I'} = C^I \in \Delta$. We want to show that $x \in (\exists e^-.\{e_C\})^{I'}$, i.e. that $(e_C^{I'}, x) \in e^{I'}$. By the definition of $e^{I'}$ in the $\mathcal{ALCOI}$ interpretation I', $(e_C^{I'}, x) \in e^{I'}$ if and only if $x \in e_C^{I'}$. But this immediately follows from the previous conclusions that: $x \in C^I$ and $e_C^{I'} = C^I$. □

Before proving the completeness of the translation of $\mathcal{ALC}^{\Omega}$ into $\mathcal{ALCOI}$, we show that, if the translation K^T of a knowledge base K in $\mathcal{ALC}^{\Omega}$ has a model in $\mathcal{ALCOI}$, then it also has a finite model.

Proposition 2. *Let K be a knowledge base in $\mathcal{ALC}^{\Omega}$ and let K^T be its translation in $\mathcal{ALCOI}$. If K^T has a model in $\mathcal{ALCOI}$, then it has a finite model.*

Proof. We prove this result by providing an alternative (but equivalent) translation $K^{T(\neg)}$ of K in the description logic $\mathcal{ALC}(\neg)$, using a single negated role $\neg e$.

$\mathcal{ALC}(\neg)$ extends $\mathcal{ALC}$ with role complement operator, where, for any role R, the role $\neg R$ is *the negation of role R*, where $(x, y) \in (\neg R)^I$ if and only if $(x, y) \notin R^I$. In the translation, we exploit $\neg e$ to capture non-membership, where $(x, y) \in (\neg e)^I$ if and only if $(x, y) \notin e^I$ (i.e., in set terms, $y \notin x$). Decidability of concept satisfiability in $\mathcal{ALC}(\neg)$ has been proved by Lutz and Sattler in [15]. The finite model property of a language with a single negated role $\neg e$ can be proved as done in [7] (Section 2) for a logic with the "window modality", by standard filtration, extended to deal with additional K-modalities (for the other roles) as in the proof in [3]. Indeed, as observed in [15], the "window operator" $\boxdot$ studied in [7] is strongly related to a negated modality, as $\boxdot \phi$ can be written as $[\neg R] \neg \phi$.

The translation $K^{T(\neg)}$ can be defined modifying K^T by replacing the concept equivalence $C \equiv \exists e^-.\{e_C\}$ with the assertions: $(\forall e.C)(e_C)$ and $(\forall (\neg e).(\neg C))(e_C)$.

One can show that any model $I = (\Delta, \cdot^I)$ di $K^{T(\neg)}$ is a model K^T in $\mathcal{ALCOI}$, and vice-versa, (considering the usual interpretation of negated roles, inverse roles and nominals). In fact, the semantic meaning of the assertion $(\forall e.C)(e_C)$ is the following:

for all $x \in \Delta$, $(e_C^I, x) \in e^I \Rightarrow x \in C^I$,

which is equivalent to the meaning of $\exists e^-.\{e_C\} \sqsubseteq C$. Also, the semantic meaning of the assertion $(\forall (\neg e).(\neg C))(e_C)$:

for all $x \in \Delta$, $(e_C^I, x) \notin e^I \Rightarrow x \notin C^I$,
i.e., for all $x \in \Delta$, $x \in C^I \Rightarrow (e_C^I, x) \in e^I$,

is equivalent to the semantic meaning of $C \sqsubseteq \exists e^-.\{e_C\}$.

We conclude the proof by observing that, if K^T has a model, it is a model of $K^{T(\neg)}$. Then, by the finite model property, $K^{T(\neg)}$ must have a finite model which is, in turn, a finite model of K^T. $\qquad\square$

To conclude our analysis we now prove the completeness of our translation.

Proposition 3 (Completeness of the translation). *The translation of an $\mathcal{ALC}^{\Omega}$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ into $\mathcal{ALCOI}$ is complete, that is, for any query F:*

$$K \models_{\mathcal{ALC}^{\Omega}} F \Rightarrow K^T \models_{\mathcal{ALCOI}} F^T.$$

Proof. We prove the completeness of the translation by contraposition. Let $K^T \not\models_{\mathcal{ALCOI}} F^T$. Then there is a model $I = \langle \Delta, \cdot^I \rangle$ of K^T in $\mathcal{ALCOI}$ such that I falsifies F.

We show that we can build a model $J = \langle \Lambda, \cdot^J \rangle$ of K in $\mathcal{ALC}^{\Omega}$. where the domain Λ is a transitive set in the universe $\mathsf{HF}^{1/2}(\mathbb{A})$ consisting of all the hereditarily finite rational hypersets built from atoms in $\mathbb{A} = \{\mathbf{a_0}, \mathbf{a_1}, \ldots\}$.

We define Λ starting from the graph[5] $G = \langle \Delta, e^I \rangle$, whose nodes are the elements of Δ and whose arcs are the pairs $(x, y) \in e^I$. Notice that, by Proposition 2, the graph

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[5] Strictly speaking the graph G introduced here is not really necessary: it is just mentioned to single out the membership relation $\in$ from e^I more clearly.

G can be assumed to be finite. Intuitively, an arc from x to y in G stands for the fact that $y \in x$.

At this point, let $\Delta_0 = \{d_1, \ldots, d_m\}$ be the elements of Δ which, in the model $I = \langle \Delta, \cdot^I \rangle$, are not in relation e^I with any other element in Δ and are non equal to the interpretation of any concept individual name e_C (that is, $d_j \in \Delta_0$ iff there is no y such that $(d_j, y) \in e^I$ and there is no concept C such that $d_j = e_C^I$). For any given $d \in \Delta$ we define the following hyperset $M(d)$:

$$M(d) = \begin{cases} \mathbf{a_k} & \text{if } d = d_k \in \Delta_0, \\ \{ M(d') \mid (d, d') \in e^I \} & \text{otherwise.} \end{cases} \tag{3}$$

Observe that, for the concepts C occurring on the l.h.s. of membership axioms, as axiom $C^T = \exists e^-.\{e_C\}$ is satisfied in the model I of K^T, it holds that $d' \in (C^T)^I$ iff $(e_C^I, d') \in e^I$. Therefore, for $d = e_C^I$, $M(d) = M(e_C^I) = \{ M(d') \mid (e_C^I, d') \in e^I \} = \{ M(d') \mid d' \in (C^T)^I \}$.

The above definition uniquely determines hypersets in $\mathsf{HF}^{1/2}(\mathbb{A})$. This follows from the fact that all finite systems of (finite) set-theoretic equations have a solution in $\mathsf{HF}^{1/2}(\mathbb{A})$. As a matter of fact, whenever the graph G is acyclic, the definition of $M(d)$ identifies a standard (recursively given) hereditarily finite set[6].

Our task now is to complete the definition of $J = \langle \Lambda, \cdot^J \rangle$ in such a way to prove that J is a model of K in $\mathcal{ALC}^\Omega$ falsifying F. The definition is completed as follows:

– $\Lambda = \{ M(d) \mid d \in \Delta \}$;
– for all $B \in N_C$, $B^J = \{ M(d) \mid d \in B^I \}$;
– for all roles $R_j \in N_R$ such that $R_j \neq e$, $R_j^J = \{ (M(d), M(d')) \mid (d, d') \in R_j^I \}$;
– for all standard name individuals $a \in N_I$ such that $a^I = d_k$, let $a^J = M(d_k) = \mathbf{a_k} \in \mathbb{A}$.

We can now prove, by induction on the structural complexity of concepts, that the following holds:

$$M(x) \in C^J \text{ if and only if } x \in (C^T)^I. \tag{4}$$

The base case for concept names, $\top$, and $\bot$ is trivial. In particular, for the case $C = B \in N_C$, by definition of J, $M(x) \in B^J$ iff $x \in B^I$. As $B^T = B$, then $x \in (B^T)^I$. Similarly for $C = \top$ and for $C = \bot$, as $\top^T = \top$, and $\bot^T = \bot$.

The inductive step in case $C = C_1 \sqcap C_2$ follows directly from the inductive hypothesis. If $M(x) \in (C_1 \sqcap C_2)^J$, then $M(x) \in C_1^J$ and $M(x) \in C_2^J$. By inductive hypothesis, $x \in (C_1^T)^I$ and $x \in (C_2^T)^I$. Hence, $x \in ((C_1 \sqcap C_2)^T)^I$. The vice-versa is proved similarly.

The cases in which $C = (\exists R.D)$ or $C = (\forall R.D)$, are also straightforward. We only consider the case $C = (\exists R.D)$. If $M(x) \in (\exists R.D)^J$, then there is a $M(d) \in \Lambda$ such that: $(M(x), M(d)) \in R^J$ and $M(d) \in D^J$. By inductive hypothesis, $d \in (D^T)^I$ and, by definition of J, $(x, d) \in R^I$. Hence, $x \in ((\exists R.D)^T)^I$.

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[6] More generally, when e^I is a well-founded relation, $M(\cdot)$ is a set-theoretic "rendering" of e^I: the so-called *Mostowski collapse* of e^I (see [13]).

For the case $C = \mathrm{Pow}(D)$, by definition of translation, we have that:

$$(C^T)^I = (\mathrm{Pow}(D^T))^I = (\forall e.D^T)^I = \{x \in \Delta \mid \forall y((x,y) \in e^I \to y \in (D^T)^I\}$$

and

$$C^J = (\mathrm{Pow}(D))^J = Pow(D^J) \cap \Lambda.$$

Consider $M(x) \in Pow(D^J) \cap \Lambda$, which is as to say that $x \in \Delta$ and $M(x) \subseteq D^J$. All the elements of $M(x)$ are of the form $M(y)$ for some $y \in \Delta$, therefore we have that:

$$\forall M(y)(M(y) \in M(x) \to M(y) \in D^J),$$

which, by definition of $M(\cdot)$ and by inductive hypothesis, means that:

$$\forall y((x,y) \in e^I \to (D^T)^I(y),$$

which proves (4) in this case.

We can now use (4) to prove that axioms and assertions in K are satisfied in J.

The cases $C \sqsubseteq D$ and $D(a)$, with C, D concepts of $\mathcal{ALC}^\Omega$ and $a \in N_I$, follow directly from (4), from the definition of $M(\cdot)$ and from the fact that $C^T \sqsubseteq D^T$ and $D^T(a)$ (respectively) are satisfied in the model I of K^T.

For each membership axiom $C \in D$ in K, we have to show that $C^J \in D^J$. As the assertion $D^T(e_C)$ is in K^T and is satisfied in I, we have $e_C^I \in (D^T)^I$. Hence, from (4), $M(e_C^I) \in D^J$. As we have seen above, $M(e_C^I) = \{M(d') \mid d' \in (C^T)^I\}$ and, again from (4), $M(e_C^I) = C^J$. Thus $C^J \in D^J$.

For each role membership axiom $(C, D) \in R$ in K, we have to show that $(C^J, R^I) \in R^J$. As the assertion $R(e_C, e_D)$ is in K^T and is satisfied in I, we have $(e_C^I, e_D^I) \in R^I$. Hence, from the definition of R^J, $(M(e_C^I), M(e_D^I)) \in R^J$. As we have seen above, $M(e_C^I) = \{M(d') \mid d' \in (C^T)^I\}$ and, from (4), $M(e_C^I) = C^J$. Similarly, $M(e_D^I) = D^J$. Thus $(C^J, D^J) \in R^J$. $\qquad\square$

As the translation of $\mathcal{ALC}^\Omega$ into $\mathcal{ALCOI}$ is polynomial (actually, linear) in the size of the knowledge base (and of the query) the following complexity result follows.

Proposition 4. *Satisfiability in $\mathcal{ALC}^\Omega$ is an* EXPTIME-*complete problem.*

The hardness comes from the EXPTIME-hardness of satisfiability in $\mathcal{ALC}$ [21]. The upper bound comes from the EXPTIME upper bound for $\mathcal{SHOI}$ [10].

5 Conclusions and related work

In this paper we have shown that the similarities between description logics and set theory can be exploited to introduce in DLs the new power-set construct and to allow for (possibly circular) membership relationships among arbitrary concepts. We have defined the description logic $\mathcal{ALC}^\Omega$, combining $\mathcal{ALC}$ with the set theory Ω, and defined its semantics whose interpretation domains are fragments of the domains of Ω-models. $\mathcal{ALC}^\Omega$ allows membership axioms among concepts as well as the power-set

construct which, up to our knowledge, has not been considered for description logics before. We have shown that an $\mathcal{ALC}^{\Omega}$ knowledge base can be polynomially translated into an $\mathcal{ALCOI}$ knowledge base. Soundness and completeness of the translation provide, besides decidability, an EXPTIME upper bound for satisfiability in $\mathcal{ALC}^{\Omega}$.

The power-set construct allows to capture in a very natural way the interactions between concepts and metaconcepts, adding to the language of $\mathcal{ALC}$ the expressivity of metamodelling. The issue of metamodelling has been analysed by Motik in [16], proving that metamodelling in $\mathcal{ALC}$-Full is already undecidable due to free mixing of logical and metalogical symbols. Two decidable semantics, a contextual π semantics and a Hilog ν-semantics, are introduced in [16] for a language extending $\mathcal{SHOIQ}$ with metamodelling, where concept names, role names and individual names are not disjoint. This possibility of using the same name in different contexts is introduced in OWL 1.1 and then in OWL 2 through *punning*[7]. As a difference, in this paper, we consider concept names, role names and individual names to be disjoint, we allow concepts (and not only concept names) to be instances of other concepts, by membership axioms, while we do not allow role names as instances.

As in [16], DeGiacomo et al. [6] and Homola et al. [11] employ an Hilog-style semantics to define $Hi(\mathcal{SHIQ})$ and $\mathcal{TH}(\mathcal{SROIQ})$, respectively. While [16] and [6] define untyped higher-order languages which, as $\mathcal{ALC}^{\Omega}$, allow a concept to be an instance of itself, [11] defines a typed higher-order extension of $\mathcal{SROIQ}$ allowing for a hierarchy of concepts, where concept names of order t can only occur as instances of concepts of order $t + 1$. In $\mathcal{TH}(\mathcal{SROIQ})$ [11] there is a strict separation between concepts and roles (as in $\mathcal{ALC}^{\Omega}$) and decidability is proved by a polynomial first-order reduction into $\mathcal{SROIQ}$, which generalizes the reduction in [8] to an arbitrary number of orders. The translation in [11] introduces axioms $A' \equiv \exists instanceOf.\{c_{A'}\}$, for each atomic concept A', axioms which are quite similar to our axiom (1), that we need for the concepts C occurring in the knowledge base on the left hand side of membership axioms.

In $Hi(\mathcal{SHIQ})$ [6], complex concept and role expressions can occur as instances of other concepts as in $\mathcal{ALC}^{\Omega}$. A polynomial translation of $Hi(\mathcal{SHIQ})$ into $\mathcal{SHIQ}$ is defined and a study of the complexity of higher-order query answering is provided.

Kubincova et al. in [14] propose a Hylog-style semantics by dropping the ordering requirement in [11] and allowing the $instanceOf$ role, with a fixed interpretation, to be used in axioms as any other role. The interpretation of role $instanceOf$ does not correspond exactly to the interpretation of e^{-} in our translation, as we do not introduce axiom (1) for all the concept names in N_C, while we introduce it for all the (possibly complex) concepts occurring as instances in some membership axiom.

Pan and Horrocks in [19] and Motz et al. in [17] define extensions of OWL DL and of $\mathcal{SHIQ}$ (respectively), based on semantics interpreting concepts as well-founded sets. In particular, [17] adds to $\mathcal{SHIQ}$ meta-modelling axioms equating individuals to concepts, without requiring that the instances of a concept need to stay in the same layer, and develop a tableau algorithm as an extension of the one for $\mathcal{SHIQ}$.

In [9] Gu introduces the language Hi(Horn-SROIQ), an extension of Horn-SROIQ which allows classes and roles to be used as individuals based on the ν-semantics [16].

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[7] https://www.w3.org/2007/OWL/wiki/Punning

ν-satisfiability and conjunctive query answering are shown to be reducible to the corresponding problems in Horn-SROIQ.

A set-theoretic approach in DLs has been adopted by Cantone et al. in [4] for determining the decidability of higher order conjunctive query answering in the description logic $\mathcal{DL}_D^{4,\times}$ (where concept and role variables may occur in queries), as well as for developing a tableau based procedure for calculating the answer sets from a $\mathcal{DL}_D^{4,\times}$ knowledge base, thus providing means for dealing with several well-known ABox reasoning tasks.

We expect that the approach of extending $\mathcal{ALC}$ with Ω can be adopted as well for more expressive DLs, which do not enjoy the finite model property. The translation proposed in this paper could indeed be extended to such logics and its soundness could be proved essentially as in Proposition 1. However, when the finite model property does not hold, there may be models of the translated knowledge base K^T containing domain elements being in the relation e with infinitely many elements, and corresponding to infinite sets. The completeness proof of Proposition 3 does not apply to this case and we leave the study of this case for future investigation.

Other possible directions for future investigation concern: the treatment of roles as individuals, which has not been considered as an option in $\mathcal{ALC}^\Omega$; restricting the semantics to well-founded sets to avoid circular definitions of sets, and exploring possible translations; translating $\mathcal{ALC}^\Omega$ into the set theory Ω, which may open to the possibility of exploiting proof methods developed for set theories in reasoning with DLs.

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